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On a Barrier Strategy for the Classical Risk Process with Constant Interest Force*

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Abstract: In this paper, we consider a problem that is due to Bruno De Finetti. The risk is described as the classical risk process with constant interest force. Dividends are paid according to a constant dividend barrier strategy. When the process reaches the barrier, all the premium income no longer goes into the surplus but is paid out as dividends to shareholders. Using the Markov property of the risk process, we obtain the explicit expression for the expectation of aggregate discounted dividends until ruin.

Keywords: classical risk process; interest force; barrier strategy; the expectation of aggregate dividends

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1 Introduction

Let $(\Omega, \Im, P)$ be a complete probability space containing all subjects defined in the following. Let $U(t)$ denote the surplus process of the company, if no dividends were paid. Then the surplus follows the following dynamics

$$\mathrm{d}U(t) = c\mathrm{d}t + U(t)\cdot r\mathrm{d}t - \mathrm{d}Z(t), \quad U(0) = u, \tag{1}$$

where $u \geq 0$ is the initial capital, $c > 0$ is the constant rate of premium income, $r \geq 0$ is the constant interest force, and $Z(t) = \sum_{k=1}^{N(t)} Z_k$, where $\{N(t),\, t \geq 0\}$ is a Poission process with intensity $\lambda > 0$, $\{Z_k\}_{k\geq 1}$ is a sequence of nonnegative i.i.d. random variables, with common distribution F and density function f.

Let T_0 denote the time of ruin, i.e.,

$$T_0 = \inf\{t \geq 0;\, U(t) < 0\} \qquad (\infty.\ \text{otherwise}) \tag{2}$$

For each $u \geq 0$, we denote by P^u the probability on (Ω, F_∞^u) generated by the surplus process with $P^u(U(0) = u) = 1$. By Theorem 32 of [1], the surplus process in (1) is a homogeneous strong Markov process possessing right continuous paths with respect to P^u.

We now consider a problem that is due to Bruno De Finetti. Suppose the above surplus model is extended by the introduction of a constant dividend barrier b. When the surplus process reaches the level b, all the income no longer goes into the surplus but is paid out

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as dividends to shareholders. Thus, when the modified surplus process attains the level b, it remains there until the next claim occurs. We denote the modified surplus process as $\tilde{U}(t)$. Let $D(t)$ denote the aggregate dividends until time t, $\delta > 0$ the force of interest for valuation, and let D denote the present value of all dividends until ruin

$$D = \int_0^T \mathrm{e}^{-\delta t}\mathrm{d}D(t), \tag{3}$$

where $T = \inf\{t \geq 0;\ \tilde{U}(t) < 0\}$ is the ruin time for the modified process. Evidently ruin will occur with certainty in this case.

For $0 \leq u \leq b$, we use the symbol $V_b(u)$ denote the expectation of D.

$$V_b(u) = E\,[\,D \mid \tilde{U}(0) = u\,]. \tag{4}$$

In section 2, we obtain the explicit expression for $V_b(u)$. The ideas and methods we used are adopted form [2, 3, 4].

2 The expectation of aggregate discounted dividends

First, we introduce some notations. We denote

$$P^u(T_0 > t, U(t) \in \mathrm{d}x) = H(t,u,x)\mathrm{d}x \tag{5}$$

and the Laplace transform of $H(t,u,x)$ as $\hat{H}_{u,x}(\delta) = \int_0^\infty \mathrm{e}^{-\delta t}H(t,u,x)\mathrm{d}t$ for $\delta \geq 0$.

Then for $t < t_x$, $H(t,u,x) = 0$, where $t_x = \frac{1}{r}\ln\frac{x+\frac{c}{r}}{u+\frac{c}{r}}$. From [2], we have

$$\hat{H}_{u,x}(\delta) = (\lambda\bar{F}(x))^{-1}f_r(\delta,u,x) \tag{6}$$

for any $u \geq 0$, $0 \leq x < u\mathrm{e}^{rt} + c\int_0^t \mathrm{e}^{rv}\mathrm{d}v$ (that is $t > t_x, x \geq 0$), where $\bar{F}(x) = 1 - F(x)$,

$$f_r(\delta,u,x) = \sum_{n=1}^{\infty} f_{r,n}(\delta,u,x). \tag{7}$$

More specifically,

$$f_{r,1}(\delta,u,x) = \frac{\lambda}{r}\left(u+\frac{c}{r}\right)^{\frac{\lambda+\delta}{r}}\left(x+\frac{c}{r}\right)^{-(1+\frac{\lambda+\delta}{r})}\bar{F}(x)I(x \geq u), \tag{8}$$

$$f_{r,n}(\delta,u,x) = \int_u^\infty \mathrm{d}x_1 \int_0^{x_1}\mathrm{d}y_1 \cdots \int_{y_{n-2}}^\infty \mathrm{d}x_{n-1}\int_0^{x_{n-1}}\mathrm{d}y_{n-1}$$

$$\int_{-\infty}^0 g_{r,n}(\delta,u,x_1,y_1,\cdots,x_{n-1},y_{n-1},x,y)\mathrm{d}y \tag{9}$$

for $r > 0$ and $n \geq 2$. Where

$$g_{r,n}(\delta,u,x_1,y_1,\cdots,x_n,y_n) = \left(\frac{\lambda}{r}\right)^n \prod_{k=1}^n \frac{(y_{k-1}+\frac{c}{r})^{\frac{\lambda+\delta}{r}}}{(x_k+\frac{c}{r})^{1+\frac{\lambda+\delta}{r}}}f(x_k - y_k) \tag{10}$$

with $y_0 = u$, and $I(\cdot)$ denoting the indicator function. For $b \geq u \geq 0$, let T_b denote the time that the surplus process (1) first reaches level b, i.e.

$$T_b = \inf\{t \geq 0;\ U(t) \geq b\}. \tag{11}$$

We first present a lemma:

Lemma 2.1 For $\delta > 0$, $0 \leq u \leq b$,

$$B(0,b|u) := E^u\left[\mathrm{e}^{-\delta T_b} I(T_b < T_0)\right] = \frac{K(u)}{K(b)}, \tag{12}$$

where

$$K(u) = h_r(\delta,u) + \left(u + \frac{c}{r}\right)^{\frac{\lambda+\delta}{r}}, \tag{13}$$

$$h_r(\delta,u) = r\big(\lambda \bar{F}(b)\big)^{-1}\left(b + \frac{c}{r}\right)^{1+\frac{\lambda+\delta}{r}} f_r(\delta,u,b). \tag{14}$$

Proof Set $t_b = \frac{1}{r}\ln\frac{b+\frac{c}{r}}{u+\frac{c}{r}}$, which is the time the surplus (1) reaches the barrier b, provided there is no claim by then. For $t \geq t_b$, we now consider $P^u(T_0 > t, T_b \in (t, t+\mathrm{d}t])$. Resort to the strong Markov property of the process, we can see by using similar arguments as [3], that for $t \geq t_b$, $P^u(T_0 > t,\ T_b \in (t, t+\mathrm{d}t])$ has density function. We denote it as $f_{u,b}(t)$ and its Laplace transform as $\hat{f}_{u,b}(\delta) = \int_{t_b}^{\infty} \mathrm{e}^{-\delta t} f_{u,b}(t)\mathrm{d}t$. Also we can get that

$$(br+c)\hat{H}_{u,b}(\delta) = (br+c)\mathrm{e}^{-(\lambda+\delta)t_b}\hat{H}_{b,b}(\delta) + (br+c)\hat{H}_{b,b}(\delta)\hat{f}_{b,b}(\delta) + \hat{f}_{u,b}(\delta). \tag{15}$$

Substituting $t_b = \frac{1}{r}\ln\frac{b+\frac{c}{r}}{u+\frac{c}{r}}$ in the above equation, we see that

$$\hat{f}_{u,b}(\delta) = \frac{(br+c)\left[\hat{H}_{u,b}(\delta) - \left(\frac{b+\frac{c}{r}}{u+\frac{c}{r}}\right)^{-\frac{\lambda+\delta}{r}}\hat{H}_{b,b}(\delta)\right]}{1 + (br+c)\hat{H}_{b,b}(\delta)}. \tag{16}$$

Therefore, we have

$$\begin{aligned}
B(0,b|u) &= E^u[\mathrm{e}^{-\delta T_b} I(T_b < T_0)] = \int_{t_b}^{\infty} \mathrm{e}^{-\delta t} f_{u,b}(t)\mathrm{d}t + \mathrm{e}^{-\delta t_b} P^u(T_b = t_b, T_0 > T_b) \\
&= \frac{(br+c)\left[\hat{H}_{u,b}(\delta) - \left(\frac{b+\frac{c}{r}}{u+\frac{c}{r}}\right)^{-\frac{\lambda+\delta}{r}}\hat{H}_{b,b}(\delta)\right]}{1 + (br+c)\hat{H}_{b,b}(\delta)} + \mathrm{e}^{-\delta\frac{1}{r}\ln\frac{b+\frac{c}{r}}{u+\frac{c}{r}}} P^u(N[0,t_0] = 0) \\
&= \frac{(br+c)\hat{H}_{u,b}(\delta) + \left(\frac{u+\frac{c}{r}}{b+\frac{c}{r}}\right)^{\frac{\lambda+\delta}{r}}}{1 + (br+c)\hat{H}_{b,b}(\delta)}.
\end{aligned}$$

Using (6), (13) and (14), we see that (12) holds true. This ends the proof.

Remark Let $r \to 0$ in (16), we can see that it corresponds to $\hat{f}_{u,a}(a)$ in [2], which has the same meaning as $\hat{f}_{u,b}(\delta)$ in our paper.

Theorem 2.1

$$V_b(u) = \frac{K(u)}{K'(b)}, \quad \text{for} \quad 0 \leq u \leq b. \tag{17}$$

Proof Since no dividends are paid unless the surplus reaches the level b before ruin occurs, we have

$$V_b(u) = E^u\big[\mathrm{e}^{-\delta T_b} I(T_b < T_0)\big] V_b(b) = B(0, b \mid u) V_b(b), \quad \text{for} \quad 0 \le u \le b. \tag{18}$$

To determine $V_b(b)$, we need a boundary condition at $u = b$. To obtain it, we compare two situations at time 0: one with initial surplus b, and the other with initial surplus $u = b - h$, $0 \le h \le b$. Then $t_0 = \frac{1}{r} \ln \frac{b+\frac{c}{r}}{b+\frac{c}{r}-h}$ is the time the surplus reaches the barrier in the second case, provided that there is no claim by then. By conditioning on the time t and the amount x of the first claim in the interval $(0, t_0)$, and noting that the dividend payments start immediately in the first case. Using strong Markov property, we see that

$$\begin{aligned} V_b(b) - V_b(b-h) &= \Big[1 - \mathrm{e}^{-(\lambda+\delta)t_0}\Big]\Big[\frac{br+c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\int_0^b V_b(b-x) f(x)\mathrm{d}x\Big] \\ &\quad - \int_0^{t_0} \mathrm{e}^{-\delta v} \int_0^{(b-h+\frac{c}{r})\mathrm{e}^{rv}-\frac{c}{r}} V_b\Big(\Big(b-h+\frac{c}{r}\Big)\mathrm{e}^{rv} - \frac{c}{r} - x\Big) f(x)\mathrm{d}x \lambda \mathrm{e}^{-\lambda v}\mathrm{d}v. \end{aligned} \tag{19}$$

Differentiating it with respect to h and then setting $h = 0$, we obtain the condition

$$\frac{\mathrm{d}V_b(u)}{\mathrm{d}u}\Big|_{u=b} = 1. \tag{20}$$

Differentiating (18) with respect to u, then setting $u = b$ and applying (20) yields

$$\frac{\partial}{\partial u} B(0, b \mid u)|_{u=b} \cdot V_b(b) = 1.$$

Hence

$$V_b(b) = \Big[\frac{\partial}{\partial u} B(0, b \mid u)|_{u=b}\Big]^{-1} = \frac{K(b)}{K'(b)}. \tag{21}$$

From (18), (21) and Lamma 2.1, we see that (17) holds true. This ends the proof.

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常利率古典风险模型下的边界分红

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摘 要: 在这篇文章中，我们考虑一个最早由Bruno De Finetti提出的问题，风险被描述为带有常利率的古典风险过程。红利按照带常数界的边界策略发放。当盈余量达到常数界时，所有的保费收入不再计入盈余，而是作为红利分发给债券持有人。利用过程的马尔可夫性，我们得到了累积期望折现分红函数的显式解。

关键词: 古典风险模型；利率；边界分红；期望折现分红